Some computations around the center problem , related to the algebra of univariate polynomials

Thesis for the M.Sc. Degree

by

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Abstract

We consider an Abel equation $y' = p(x)y^2 + q(x)y^3$ with $p(x)$, $q(x)$ are polynomials in x. A center condition for this equation (closely related to the classical center condition for polynomial vector fields on the plane) is that $y_0 = y(0) \equiv y(1)$ for any solution $y(x)$. This condition is given by vanishing of all the Taylor coefficients $v_k(1)$ in the development $y(x) = y_0 + \sum_{k=2}^{\infty} v_k(x) y_0^k$. A new basis for the ideals $I_k = \{v_2, \ldots, v_k\}$ has been produced in [2], defined by a linear recurrence relation. In this article we discuss some questions concerning the behavior of the ideals I_k and some other questions, closely related to this subject.

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Chapter 1

Research objectives

1.1 Introduction

In [7] H.Poincaré defined the notion of a center for a real vector field on the plane

$$\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases}$$

as an isolated singularity surrounded by closed integral curves. He showed (see [8]) that a necessary and sufficient condition for a polynomial vector field (i.e. $f(x,y) = P(x,y)$, $g(x,y) = Q(x,y)$ are polynomials in x, y) with a singular point with pure imaginary eigenvalues, to have a center at this point is the annihilation of an infinite number of polynomials in the coefficients of the vector field. The problem of explicitly finding a finite basis for these algebraic conditions (the problem of the center), was solved in the case of quadratic vector fields by the successive contributions of H.Dulac, W.Kapteyn, N.Bautin, N.Sakharnikov, L.Belyustina, K.Sibirsky and others (see e.g. [1], [9]). The complete conditions on $P(x,y)$, $Q(x,y)$ of degrees higher then 2 under which the system has a center are still unknown.

1.2 Description of the problem

1.2.1 The center problem

We will consider the following formulation of the center problem: Let $P(x, y)$, $Q(x, y)$ be polynomials in x, y of degree d. Consider the system of differential equations

$$\begin{cases} \dot{x} = -y + P(x, y) \\ \dot{y} = x + Q(x, y) \end{cases} \tag{1.1}$$

We will say that a solution $x(t)$, $y(t)$ of (1.1) is closed if it is defined in the interval $[0, t_0]$ and $x(0) = x(t_0)$, $y(0) = y(t_0)$. We will say that the system (1) has a center at 0 if all the solutions around zero are closed. Then the general problem is: under what conditions on P, Q the system (1.1) has a center at zero?

1.2.2 Reduction to the Abel equation

It was shown in [4] that one can reduce the system (1.1) with homogeneous P, Q of degree d to the Abel equation

$$y' = p(x)y^2 + q(x)y^3 \tag{1.2}$$

where $p(x)$, $q(x)$ are polynomials in $\sin x$, $\cos x$ of degrees depending only on d. Then (1.1) has a center if and only if (1.2) has periodic solutions on $[0, 2\pi]$, i.e. solutions $y = y(x)$ satisfying $y(0) = y(2\pi)$.

1.2.3 Classical approach to the study of the Abel equation

We will look for solutions of (1.2) in the form

$$y(x, y_0) = y_0 + \sum_{k=2}^{\infty} v_k(x, \lambda) y_0^k,$$

where $y(0, y_0) = y_0$, and v_k turn out to be polynomials both in x and λ, where $\lambda = (\lambda_1, \lambda_2, \ldots)$ is the (finite) set of the coefficients of p, q. Shortly we will write $v_k(x)$.

Then $y(2\pi) = y(2\pi, y_0) = y_0 + \sum_{k=2}^{\infty} v_k(2\pi) y_0^k$ and hence the condition $y(2\pi) \equiv y(0)$ is equivalent to $v_k(2\pi) = 0$ for $k = 2, 3, \ldots \infty$,

Consider an ideal $J = \{v_2(2\pi), v_3(2\pi), \ldots v_k(2\pi), \ldots\} \subseteq \mathbb{C}[\lambda]$. By Hilbert Basis theorem there exists $d_0 < \infty$, s.t. $J = \{v_2(2\pi), v_3(2\pi), \ldots v_{d_0}(2\pi)\}$. After determination of d_0 the general problem will be solved, since we get finite number of conditions on λ, which define the set of p, q having all the solutions closed . The problem is that the Hilbert theorem does not allow us to define d_0 constructively.

1.2.4 Modified approach to the study of the Abel equation

Let us study instead of $J \subseteq \mathbb{C}[\lambda]$ the polynomials ideal $I \subseteq \mathbb{C}[\lambda, x]$, $I = \{v_2(x), v_3(x), \ldots v_k(x), \ldots\} = \bigcup_{k=2}^{\infty} I_k$, where $I_k = \{v_2(x), v_3(x), \ldots v_k(x)\}$. The classical problem is to find conditions on p, q, under which $x = 2\pi$ is a common zero of all I_k.

Our **generalized center problem** consists of the following:

a) Study the behavior of I_k as the ideals of univariate polynomials in x, i.e.
i. For given p, q find zeroes in x of I_k, $k = 2, \ldots$ and of $I = \bigcup_{k=2}^{\infty} I_k$.
ii. For a given set of numbers find conditions on p, q, under which these numbers are common zeroes of I.

b) Find the stabilization moment of the set of common zeroes, i.e.
i. For given p, q find d, for which the set of common zeroes of I is equal to the set of common zeroes of I_d.
ii. For given set of common zeroes of I find d, for which it is equal to the set of common zeroes of I_d. Under which conditions on p, q is it possible?

c) For given p, q find d, for which $I = I_d$ (Bautin's problem).

1.2.5 Main recurrence relations

In what follows we shall study Abel equation (1.2) with p, q the usual polynomials in x instead of trigonometric ones. In this case we say that the equation (1.2) defines a center if $y(1, y_0) \equiv y_0$. Although this property does not correspond to the initial problem (1.1), it presents an interest by itself and it has been studied in [5], [6] and in many others articles. Our main goal is to study the generalized center problem for this case, our first goal is to study part a) of it.

One can easily show (see e.g. [2]) that $v_k(x)$ satisfy recurrence relations

$$\begin{cases} v_0(x) \equiv 0 \\ v_1(x) \equiv 1 \\ v_n'(x) = p(x) \sum\limits_{i+j=n} v_i(x)v_j(x) + q(x) \sum\limits_{i+j+k=n} v_i(x)v_j(x)v_k(x), \ n \geq 2 \end{cases} \tag{1.3}$$

It was shown in [2] that in fact this recurrence relations can be linearized, i.e. the same ideals I_k's are generated by $\{\psi_1, \ldots \psi_k\}$, where $\psi_k(x)$ satisfy linear recurrence relations

$$\boxed{\begin{cases} \psi_0(x) \equiv 0 \\ \psi_1(x) \equiv 1 \\ \psi_n'(x) = -(n-1)\psi_{n-1}(x)p(x) - (n-2)\psi_{n-2}(x)q(x), \ n \geq 2 \end{cases}} \tag{1.4}$$

which are much more convenient then (1.3). We call (1.4) **the main recurrence relation** for the main problem.

1.2.6 The first model problem

Let us state an auxiliary problem:

The first model problem : Given $p(x)$, $Q_0(x)$. Define $Q_{k+1}(x)$ by recurrence relation $Q_{k+1}(x) = \int_0^x p(t)Q_k(t)dt$, $k \geq 0$. Study the generalized center problem for the ideals $I_k = \{Q_0(x), \ldots, Q_k(x)\}$.

As it was shown in [2], the same ideal is generated by the polynomials P_i, $i = 0, \ldots, k$, where $P_i(x) = \int_0^x P^i(t)q(t)dt$, $P(x) = \int_0^x p(t)dt$, $q(x) = Q_0'(x)$.

Hopefully this problem can help us to study the main problem (1.4). It allows for "analytic" solutions (through generating functions).

For the main problem the first few ideals I_k are very similar to the first few ideals of the first model problem , but starting with the I_6 essentially nonlinear equations with respect to $Q(x) = \int_0^x q(t)dt$ appear. This fact presents the main difficulty in analysis of the problem (1.4).

1.2.7 Main conjecture and known results

The following conjecture for the main problem (1.4) has been proposed in [2]:

$I = \bigcup_{k=1}^{\infty} I_k$ has zeroes $a_1, \ldots a_k$, $a_1 = 0$ if and only if

$P(x) = \int_0^x p(t)dt = \tilde{P}(W(x))$, $Q(x) = \int_0^x q(t)dt = \tilde{Q}(W(x))$,

where $W(x) = \prod_{i=1}^{k}(x - a_i)$,

$\tilde{P}$, $\tilde{Q}$ are some polynomials without free terms.

Exactly the same conjecture can be stated for the first model problem, with $Q = Q_0$. One can easily show (see [2]) that these conditions are sufficient for zeroes of W to be common zeroes of I in each of these problems. It is not clear yet if these conditions are also necessary.

The following particular results are known:

1) The conjecture is true for $P(x)$, $Q(x)$ up to degree 3 and for some cases of degree 4 (see [2]).

2) For the first model problem if $P(x) = W(x) = \prod_{i=1}^{k}(x - a_i)$, then $\bigcup_{k=1}^{\infty} I_k$ has zeroes $a_1, \ldots a_k$ if and only if $Q_0(x) = \tilde{Q}_0(W(x))$ (see [3]).

3) For the first model problem combinatorial estimation of the I_k's stabilization moment is obtained (see [10], [11]).

1.2.8 Results

We present the following results:

a) Some remarks, connected to the first model problem. They can be useful as a tool for an estimation of the number of surviving zeroes (chapter 2).

b) We have obtained the number of conditions, which should be checked in order to say, that the hypothesis for the first model problem is true, and some remarks about sufficiency of this number (chapter 3).

c) Maximal number of zeroes of I for the recurrence relation (1.4) is obtained (chapter 4).

d) Verification of the main conjecture 2.7 for the main problem (1.4) with the degrees of P , Q up to 4 and in some cases of higher degrees . It was done using computer symbolic calculations with some convenient representation of P and Q (chapter 5).

Chapter 2

Some remarks around the first model problem.

2.1 “The zero model problem.”

Consider so called **“the zero model problem” :** Given $\phi_0(x)$. Define $\phi_{k+1}(x)$ by recurrence relation

$$\phi_{k+1}(x) = \int_0^x \phi_k(t)dt, \qquad k \geq 0.$$

Study the generalized center problem for the ideals $I_k = \{\phi_0(x), \dots, \phi_k(x)\}$.

Claim: *All $\phi_k(x)$ may have the only one common zero – 0.*

This consideration may be useful as a method for the estimation of the number of zeroes.

Proof:

1) Let $\phi_d(x)$ has roots $a_1 \neq a_2$ and they are common zeroes of $\phi_0(x), \dots, \phi_d(x)$. Then

$$\phi_d(a_1) = 0, \quad \phi_d'(a_1) = 0 \quad , \dots , \quad \phi_d^{(d)}(a_1) = 0,$$

therefore $\phi_d(x) = (x - a_1)^{d+1} S_1(x)$.

Similarly $\phi_d(x) = (x - a_2)^{d+1} S_2(x)$, hence

$$\phi_d(x) = (x - a_1)^{d+1} (x - a_2)^{d+1} S(x),$$

and therefore $\deg \phi_d(x) \geq 2d+2$. But $\deg \phi_d(x) = \deg \phi_0(x) + d = 2d$. Contradiction.

2) Therefore the ideal may have only one zero. Obviously if $\phi_0(0) = 0$, then 0 will be such common root. Assume now that $\phi_0(0) \neq 0$, but a is a common root of all $\phi_k(x)$.

If $\phi_0(x) = (x-a)^k S_0(x)$, $S_0(x)$ is a polynomial, then obviously $\phi_j(x) = (x-a)^{k+j} S_j(x)$ with $S_j(x)$ - a polynomial. But also $\phi_j(x) = x^j \tilde{S}_j(x)$, and hence $\phi_j(x) = x^j (x-a)^{k+j} \tilde{\tilde{S}}_j(x)$, therefore $\deg \phi_d(x) = j + (k+j) + \deg \tilde{\tilde{S}}_j(x) \geq 2j + k$. But if $\deg \phi_0(x) = d$, then $\deg \phi_d(x) = d + j$. Taking $j = d - k + 1$, we get contradiction.

2.2 Some remarks around the first model problem.

Using the same technique as in the proof of Claim, we can make the following remarks:

a) Consider the first model problem $Q_j(x) = \displaystyle\int_0^x p(t) Q_{j-1}(t) dt$ with $\deg Q_0 = d$, $\deg p = m$. Let a, 0 be common zeroes of all $Q_j(x)$ and $p(0) \neq 0$, $p(a) \neq 0$. Then

$$Q_j(x) = (x-a)^{k+j} x^{l+j} S_j(x), \text{ where } S_j(0) \neq 0, S_j(a) \neq 0.$$

So, the degree of $S_j(x)$ is $(d-k-l) + j(m-1)$ and it grows on $m-1$ on each step $(j \mapsto j+1)$. Then we get

$$p(x) S_{j-1}(x) = (k+j) x S_j(x) + (l+j)(x-a) S_j(x) + (x-a) x S_j'(x).$$

As the converse recursion this formula may be useful for determination which Q_0 corresponds to the given Q_j, $p(x)$. Since this formula was deduced in assumption that $a, 0$ are common zeroes of all $Q_j(x)$, this formula has to lead to the contradiction for some $Q_j(x)$ and $p(x)$.

b) Similarly if $p(0) = 0$, $p(a) \neq 0$ we obtain

$$Q_j(x) = (x-a)^{k+j} x^{l+2j} S_j(x), \text{ where } S_j(0) \neq 0, S_j(a) \neq 0.$$

Then the degree of $S_j(x)$ is $(d-k-l)+j(m-2)$ and it grows on $m-2$ on each step. The "converse recursion" formula in this case is

$$p(x)S_{j-1}(x) = (k+j)x^2S_j(x) + (l+2j)(x-a)xS_j(x) + (x-a)x^2S_j'(x).$$

c) Similar recursive formulae are obtained for the case when $p(0) \neq 0$, $p(a) = 0$ (the growing on $m-2$) and for the case when $p(0) = 0$, $p(a) = 0$ (the growing on $m-3$).

Chapter 3

The lower bound for the number of conditions, which are necessary for the proof of the conjecture for the first model problem.

3.1 Introduction.

We are interested in proving the main conjecture 1.2.7. for the first model problem. The following theorem was proved by M. Briskin in [2]:

Theorem 3.0. *Let*

$$P(x) = W(x) = \prod_{i=1}^{\nu}(x - a_i), \quad a_1 = 0, \quad a_i \neq a_j$$

Let $Q(t)$ be a polynomial of degree $m\nu + \alpha$, $0 \leq \alpha \leq \nu - 1$. Denote a "moment" $V_l(x) = \int_0^a P^l(t)q(t)dt$. If $a_1, \dots, a_\nu$ are common zeroes of $m+1$ consecutive moments $V_\ell(x), V_{\ell+1}(x), \dots, V_{\ell+m}(x)$ for some $\ell \geq 0$, then $\alpha = 0$ and $Q(x)$ can be represented as $Q(x) = \tilde{Q}(W)$, for a certain polynomial $\tilde{Q}$ of degree m without free term. If we know a priori that $\alpha = 0$ then vanishing of only m moments $V_\ell, \dots, V_{\ell+m-1}$ implies $Q = \tilde{Q}(W)$.

The results of this chapter give a generalization of this result and the approach to prove it using completely different arguments.

More precisely, theorem 3.3 proves that $m+1$ is necessarily for the statement of the theorem 3.0 , i.e. it is impossible to decrease the number of vanishing moments. Theorems 3.4 and 3.5 generalize this result for

$$W(x) = \prod_{i=1}^{\nu}(x-a_i)^{k_i}, \quad a_1 = 0, \quad a_i \neq a_j$$

and

$$P(x) = \gamma_n W^n + \gamma_{n-1} W^{n-1} + \cdots + \gamma_0.$$

Theorems 3.1 and 3.2 , which are particular cases of theorem 3.3 , describe in all details the method introduced in section 3.2 and give a connection between a composition of polynomials and linear algebra.

3.2 A convenient representation of P and Q and algebra of compositions of polynomials.

Assume we are interested in the checking if numbers 0, a are common zeroes of our ideal $I = \bigcup_{k=0}^{\infty} I_k$. Let $R(x)$ be an arbitrary polynomial of degree n. Consider $W(x) = x(x-a)$ - polynomial of the second degree. Notice, that the derivative of W is a polynomial of the first degree $W'(x)$, the polynomial $W(x)W'(x)$ has the third degree and so on. Generally, polynomials $W(x)^k$ have degree $2k$ and polynomials $W(x)^k\ W'(x)$ have degree $2k+1$. Therefore they are linearly independent and form a basis of $\mathbb{C}[x]$. So, one can uniquely represent any polynomial $R(x)$ as a linear combination of polynomials $W(x)^k$ and $W(x)^k W'(x)$. Hence the polynomial $R(x)$ of the degree $2k$ or $2k+1$ we will write in the form

$$R(x) = W(x)^k\ (\alpha_k W(x)' + \beta_k) + W(x)^{k-1}\ (\alpha_{k-1} W(x)' + \beta_{k-1}) + ... + (\alpha_0 W(x)' + \beta_0),$$

or simply

$$R(x) = W^k(\alpha_k W' + \beta_k) + W^{k-1}(\alpha_{k-1} W' + \beta_{k-1}) + ... + (\alpha_0 W' + \beta_0).$$

In general, if $W(x) = x(x-a_2)\dots(x-a_k)$, $\deg W(x) = k$, then any polynomial $R(x)$ can be uniquely represented in the form

$$R(x) = W^m(c_m^1 W' + c_m^2 W'' + \ldots + c_m^k W^{(k)}) + \ldots + (c_0^1 W' + c_0^2 W'' + \ldots + c_0^k W^{(k)}),$$

(where, of course, $W^{(k)}$ is a constant).

Now we see, that to prove that $\int_0^x R(t)dt$ is a composition with $W(x)$ we have to prove that $c_j^i = 0$ for $i \geq 2$, $j = 0, \ldots, m$.

Theorems 3.1 and 3.2 demonstrate the usefulness of this representation. Also it will be widely used for the verification of the main conjecture (see chapter 5) , where it seems to be the most convenient way of proving.

3.3 The lower bound for the number of conditions, which are necessary for the proof of the conjecture.

Theorem 3.1. *Let $P(x) = W(x) = x(x-a)$, $a \neq 0$.*
Let Q be a polynomial of degree $2(m+1) - \alpha$, $\alpha = 0, 1$.
Denote $V_j = \int_0^a P^j(t)q(t)dt$. Then the minimal necessary number $n+1$ of conditions $V_{j_1} = 0, \ldots, V_{j_{n+1}} = 0$ (for different but not necessary consecutive j_k, i.e. not necessary $j_{k+1} = j_k + 1$) for the conclusion $Q = \tilde{Q}(W)$ for a certain polynomial without free term is $m+1$ and this result does not depend on a. For any number of conditions which is less than $m+1$ there exists Q unrepresentable as a composition with W, for which all V_j listed above are zeroes.

Proof of the theorem 3.1.

Assume that $\int_0^a P^j q = 0$ for $j = j_1, j_2, \ldots, j_{n+1}$. Notice, that for

$$q(x) = W^m(\alpha_m W' + \beta_m) + W^{m-1}(\alpha_{m-1} W' + \beta_{m-1}) + \ldots + (\alpha_0 W' + \beta_0)$$

we get

$$\int_0^a W^j q = \beta_m \int_0^a W^{m+j} + \beta_{m-1} \int_0^a W^{m+j-1} + \ldots + \beta_0 \int_0^a W^j.$$

Our goal is to find minimal n, such that the system

$$\begin{cases} \displaystyle\int_0^a W^{j_1} q & = 0 \\ \displaystyle\int_0^a W^{j_2} q & = 0 \\ \quad\vdots & \quad\vdots \\ \displaystyle\int_0^a W^{j_{n+1}} q & = 0 \end{cases}$$

will have the unique solution $\beta_0 = \beta_1 = \ldots = \beta_m = 0$.

Denote $W_j = \displaystyle\int_0^a W(t)^j dt$. Then this system can be rewritten in the form

$$\begin{cases} \beta_m W_{m+j_1} & + & \beta_{m-1} W_{m+j_1-1} & + & \cdots & + & \beta_0 W_{j_1} & = 0 \\ \beta_m W_{m+j_2} & + & \beta_{m-1} W_{m+j_2-1} & + & \cdots & + & \beta_0 W_{j_2} & = 0 \\ \vdots & & \vdots & & \ddots & & \vdots & \vdots \\ \beta_m W_{m+j_{n+1}} & + & \beta_{m-1} W_{m+j_{n+1}-1} & + & \cdots & + & \beta_0 W_{j_{n+1}} & = 0 \end{cases}$$

i.e.

$$\begin{pmatrix} W_{j_1} & W_{j_1+1} & \cdots & W_{j_1+m} \\ W_{j_2} & W_{j_2+1} & \cdots & W_{j_2+m} \\ \vdots & \vdots & \ddots & \vdots \\ W_{j_{n+1}} & W_{j_{n+1}+1} & \cdots & W_{j_{n+1}+m} \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_m \end{pmatrix} = 0.$$

Now we see that the linear homogeneous system with the $(n+1)\times(m+1)$ matrix has the unique zero solution only if $n \geq m$ and det of the squarmatrix is nonzero. For $n < m$ the system always has nonzero solution. The theorem is proved.

Remarks:

a)To prove the conjecture for the first model problem it is enough to show that for at least one special sequence $j_1, \ldots, j_{m+1}$ the

$$\det \begin{pmatrix} W_{j_1} & W_{j_1+1} & \cdots & W_{j_1+m} \\ W_{j_2} & W_{j_2+1} & \cdots & W_{j_2+m} \\ \vdots & \vdots & \ddots & \vdots \\ W_{j_{n+1}} & W_{j_{n+1}+1} & \cdots & W_{j_{n+1}+m} \end{pmatrix} \neq 0.$$

b) Let $W_{k,n} = \displaystyle\int_0^a W(t)^k (W'(t))^n dt$. Then the formula for $W_{k,n}$ is obtained:

$$\begin{aligned} W_{k,2n+1} &= 0 \\ W_{k,2n} &= \frac{(-1)^k a^{2k+2n+1} k!(2n-1)!!}{2^k(2k+2n+1)!!} \end{aligned}$$

c) From [2] follows that

$$D_{k,m} = \begin{vmatrix} W_k & W_{k+1} & \cdots & W_{k+m} \\ \vdots & \vdots & \ddots & \vdots \\ W_{k+m} & W_{k+m+1} & \cdots & W_{k+2m} \end{vmatrix} \neq 0.$$

For instance,

$$D_{0,m} = \sum_{\sigma \in S_{0,m}} (-1)^\sigma W_{\sigma(0)} W_{\sigma(1)+1} \cdots W_{\sigma(m)+m} =$$

$$= a^{2m^2+2m+3} \sum_{\sigma \in S_{0,m}} (-1)^\sigma \frac{\sigma(0)!(\sigma(1)+1)! \cdots (\sigma(m)+m)!}{2^{m(m+1)}(2\sigma(0))!! \cdots (2\sigma(m)+2m+1)!!} \neq 0$$

for any m.

d) Using "Mathematica" we have computed some $D_{0,k}$ and received very nice expressions:

$D_{0,1}^{-1} = 2^2 \quad 3^2 \quad 5$

$D_{0,2}^{-1} = 2^4 \quad 3^4 \quad 5^3 \quad 7^2$

$D_{0,3}^{-1} = 2^8 \quad 3^6 \quad 5^3 \quad 7^4 \quad 11^2 \quad 13$

$D_{0,4}^{-1} = 2^{10} \quad 3^{10} \quad 5^5 \quad 7^4 \quad 11^4 \quad 13^3 \quad 17$

$D_{0,5}^{-1} = 2^{14} \quad 3^{11} \quad 5^5 \quad 7^5 \quad 11^6 \quad 13^5 \quad 17^3 \quad 19^2$

$D_{0,6}^{-1} = 2^{18} \quad 3^{11} \quad 5^7 \quad 7^7 \quad 11^6 \quad 13^7 \quad 17^5 \quad 19^4 \quad 23^2$

$D_{0,7}^{-1} = 2^{24} \quad 3^{17} \quad 5^{11} \quad 7^7 \quad 11^6 \quad 13^7 \quad 17^7 \quad 19^6 \quad 23^4 \quad 29$

$D_{0,8}^{-1} = 2^{26} \quad 3^{22} \quad 5^{13} \quad 7^7 \quad 11^7 \quad 13^7 \quad 17^9 \quad 19^8 \quad 23^6 \quad 29^3 \quad 31^2$

e) Deducing (b) we have obtained for nothing the following very nice combinatorial identity:

$$\sum_{i=0}^{n}(-1)^i \left(\begin{array}{c} i \\ n \end{array} \right) \frac{(k+i)!2^i}{(2k+2i+1)!!} = \frac{k!(2n-1)!!}{(2k+2n+1)!!}.$$

Theorem 3.2. *Let* $P(x) = W(x) = x(x-a)(x-b)$, $a \neq b$, $a, b \neq 0$.
Let Q *be a polynomial of degree* $3(m+1) - \alpha$, $\alpha = 0, 1, 2$.
Denote $V_j(x) = \int_0^x P^j(t)q(t)dt$. *Then the minimal necessary number* $n+1$ *of conditions* $V_{j_1}(a) = 0, V_{j_1}(b) = 0, \ldots, V_{j_{n+1}}(a) = 0, V_{j_{n+1}}(b) = 0$ *(for different but not not necessary consecutive* j_k*) for the conclusion* $Q = \tilde{Q}(W)$ *for a certain polynomial* $\tilde{Q}$ *without free term is* $m+1$ *. For any number of conditions which is less than* $m+1$ *there exists* Q *unrepresentable as a composition with* W*, for which all* $V_j(a)$*,* $V_j(b)$ *listed above are zeroes.*

Proof of the theorem 3.2.

Assume that $\int_0^a P^j q = 0$, $\int_0^b P^j q = 0$ for $j = j_1, j_2, \ldots, j_{n+1}$. For

$$q(x) = W^m(\alpha_m W' + \beta_m W'' + \gamma_m) + W^{m-1}(\alpha_{m-1}W' + \beta_{m-1}W'' + \gamma_{m-1}) + \ldots$$
$$+(\alpha_0 W' + \beta_0 W'' + \gamma_0)$$

we get

$$\int_0^a W^j q = \beta_m \int_0^a W^{m+j}W'' + \beta_{m-1}\int_0^a W^{m+j-1}W'' + \ldots + \beta_0 \int_0^a W^j W'' +$$
$$+\gamma_m \int_0^a W^{m+j} + \gamma_{m-1}\int_0^a W^{m+j-1} + \ldots + \gamma_0 \int_0^a W^j,$$

since the coefficients by α are of the form $\int_0^a W^k W'$ and hence they are equal to zero. And the similar expressions hold for b instead of a.

Our goal is to find minimal n, such that the system

$$\begin{cases} \int_0^a W^{j_1} q &= 0 \quad , \int_0^b W^{j_1} q &= 0 \\ \int_0^a W^{j_2} q &= 0 \quad , \int_0^b W^{j_2} q &= 0 \\ \vdots & \vdots \quad\quad \vdots & \vdots \\ \int_0^a W^{j_{n+1}} q &= 0 \quad , \int_0^b W^{j_{n+1}} q &= 0 \end{cases}$$

will have the unique solution $\beta_0 = \beta_1 = \ldots = \beta_m = \gamma_0 = \gamma_1 = \ldots = \gamma_m = 0$.

This system is equivalent to the following:

$$\begin{cases} \sum_{i=0}^m \beta_i \int_0^a W^{j_s+i}W'' + \sum_{i=0}^m \gamma_i \int_0^a W^{j_s+i} = 0, s = 1, \ldots, n+1 \\ \sum_{i=0}^m \beta_i \int_0^b W^{j_s+i}W'' + \sum_{i=0}^m \gamma_i \int_0^b W^{j_s+i} = 0, s = 1, \ldots, n+1 \end{cases}$$

Denote

$$F_{k,i}(x) = \int_0^x W(t)^{k+i} W''(t)dt$$

$$G_{k,i}(x) = \int_0^x W(t)^{k+i} dt.$$

Then we get the following system:

$$\left\{\begin{array}{ccccc}
\beta_0 F_{j_1,0}(a) & +\beta_1 F_{j_1,1}(a) & +\cdots & +\beta_m F_{j_1,m}(a) & + \\
+\gamma_0 G_{j_1,0}(a) & +\gamma_1 G_{j_1,1}(a) & +\cdots & +\gamma_m G_{j_1,m}(a) & =0 \\
\\
\beta_0 F_{j_2,0}(a) & +\beta_1 F_{j_2,1}(a) & +\cdots & +\beta_m F_{j_2,m}(a) & + \\
+\gamma_0 G_{j_2,0}(a) & +\gamma_1 G_{j_2,1}(a) & +\cdots & +\gamma_m G_{j_2,m}(a) & =0 \\
\\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\\
\beta_0 F_{j_{n+1},0}(a) & +\beta_1 F_{j_{n+1},1}(a) & +\cdots & +\beta_m F_{j_{n+1},m}(a) & + \\
+\gamma_0 G_{j_{n+1},0}(a) & +\gamma_1 G_{j_{n+1},1}(a) & +\cdots & +\gamma_m G_{j_{n+1},m}(a) & =0 \\
\beta_0 F_{j_1,0}(b) & +\beta_1 F_{j_1,1}(b) & +\cdots & +\beta_m F_{j_1,m}(b) & + \\
+\gamma_0 G_{j_1,0}(b) & +\gamma_1 G_{j_1,1}(b) & +\cdots & +\gamma_m G_{j_1,m}(b) & =0 \\
\\
\beta_0 F_{j_2,0}(b) & +\beta_1 F_{j_2,1}(b) & +\cdots & +\beta_m F_{j_2,m}(b) & + \\
+\gamma_0 G_{j_2,0}(b) & +\gamma_1 G_{j_2,1}(b) & +\cdots & +\gamma_m G_{j_2,m}(b) & =0 \\
\\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\\
\beta_0 F_{j_{n+1},0}(b) & +\beta_1 F_{j_{n+1},1}(b) & +\cdots & +\beta_m F_{j_{n+1},m}(b) & + \\
+\gamma_0 G_{j_{n+1},0}(b) & +\gamma_1 G_{j_{n+1},1}(b) & +\cdots & +\gamma_m G_{j_{n+1},m}(b) & =0
\end{array}\right.$$

i.e.

$$\left\{\begin{array}{l} F(a)\beta + G(a)\gamma = 0 \\ F(b)\beta + G(b)\gamma = 0 \end{array}\right. ,$$

where

$$\beta = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_m \end{pmatrix}, \gamma = \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \vdots \\ \gamma_m \end{pmatrix},$$

$$F(a) = \begin{pmatrix} F_{j_1,0}(a) & F_{j_1,1}(a) & \cdots & F_{j_1,m}(a) \\ F_{j_2,0}(a) & F_{j_2,1}(a) & \cdots & F_{j_2,m}(a) \\ \vdots & \vdots & \ddots & \vdots \\ F_{j_{n+1},0}(a) & F_{j_{n+1},1}(a) & \cdots & F_{j_{n+1},m}(a) \end{pmatrix},$$

$$G(a) = \begin{pmatrix} G_{j_1,0}(a) & G_{j_1,1}(a) & \cdots & G_{j_1,m}(a) \\ G_{j_2,0}(a) & G_{j_2,1}(a) & \cdots & G_{j_2,m}(a) \\ \vdots & \vdots & \ddots & \vdots \\ G_{j_{n+1},0}(a) & G_{j_{n+1},1}(a) & \cdots & G_{j_{n+1},m}(a) \end{pmatrix},$$

$$F(b) = \begin{pmatrix} F_{j_1,0}(b) & F_{j_1,1}(b) & \cdots & F_{j_1,m}(b) \\ F_{j_2,0}(b) & F_{j_2,1}(b) & \cdots & F_{j_2,m}(b) \\ \vdots & \vdots & \ddots & \vdots \\ F_{j_{n+1},0}(b) & F_{j_{n+1},1}(b) & \cdots & F_{j_{n+1},m}(b) \end{pmatrix},$$

$$G(b) = \begin{pmatrix} G_{j_1,0}(b) & G_{j_1,1}(b) & \cdots & G_{j_1,m}(b) \\ G_{j_2,0}(b) & G_{j_2,1}(b) & \cdots & G_{j_2,m}(b) \\ \vdots & \vdots & \ddots & \vdots \\ G_{j_{n+1},0}(b) & G_{j_{n+1},1}(b) & \cdots & G_{j_{n+1},m}(b) \end{pmatrix}.$$

Now we see that the linear homogeneous system with the $(2n+2) \times (2m+2)$ matrix has the unique zero solution only if $n \geq m$ and det of the squarmatrix is nonzero. For $n < m$ the system always has nonzero solution. The theorem is proved.

Remarks:

a) For the consecutive set of indexes $j_l, \ldots, j_{m+l}$, starting from l, we get $\det H$ is a homogeneous polynomial in a, b of degree $3[(n+1)(2n+1)+l(2n+2)]$.

b) Using "Mathematica" we computed $\det H_{0,1} = \frac{1}{20}(ab(a-b))^6$, $\det H_{0,2} = \frac{1}{520}(ab(a-b))^{15}$ and so on.

c) Obviously $\det H$ does not depend on the change $a \mapsto a - b$.

d) Conjecture: $\det H_{0,n} = Const[ab(a-b)]^{(n+1)(2n+1)}$.

Theorem 3.3. *Let*

$$P(x) = W(x) = \prod_{i=1}^{\nu}(x - a_i), \quad a_1 = 0, \quad a_i \neq a_j \quad for \quad i \neq j.$$

Let $Q(t)$ be a polynomial of degree $\nu(m+1) - \alpha$, $\alpha = 0, \ldots \nu - 1$. Denote $V_j(x) = \int_{a_1}^{x} P^j(t)q(t)dt$. Then the minimal necessary number $n+1$ of conditions $V_j(x) = 0$ for $x = a_2, \ldots, a_\nu$ (not necessary consecutive) for the conclusion $Q = \tilde{Q}(W)$ for a certain polynomial $\tilde{Q}$ without free term is $m+1$. For any number of conditions which is less than $m+1$ there exists Q unrepresentable as a composition with W, for which all $V_j(a_2) = \ldots = V_j(a_\nu)$ listed above are zeroes.

Proof of the theorem 3.3.

Assume that $\int_{a_1}^{a_k} P^j q = 0$, for $j = j_1, j_2, \ldots, j_{n+1}$, $k = 2, \ldots, \nu$. For

$$q(x) = W^m(c_m^1 W' + c_m^2 W'' + \ldots + c_m^\nu W^{(\nu)}) + \ldots + (c_0^1 W' + c_0^2 W'' + \ldots + c_0^\nu W^{(\nu)})$$

we get

$$\int_{a_1}^{a_l} W^k q = \sum_{i=0}^{m} \left(\sum_{j=2}^{\nu} c_i^j \int_{a_1}^{a_l} W^{k+i} W^{(j)} \right)$$

since the coefficients of the form $\int_0^a W^k W'$ are equal to zero.

Our goal is to find minimal n, such that the system

$$\begin{cases} \int_{a_1}^{a_2} W^{j_1} q &= 0 \quad ,\dots, \quad \int_{a_1}^{a_\nu} W^{j_1} q &= 0 \\ \int_{a_1}^{a_2} W^{j_2} q &= 0 \quad ,\dots, \quad \int_{a_1}^{a_\nu} W^{j_2} q &= 0 \\ \vdots \qquad\qquad \vdots & \quad \ddots \qquad\qquad \vdots \qquad\qquad \vdots \\ \int_{a_1}^{a_2} W^{j_{n+1}} q &= 0 \quad ,\dots, \quad \int_{a_1}^{a_\nu} W^{j_{n+1}} q &= 0 \end{cases}$$

will have the unique solution $\{c_i^j\}_{i=0,\dots,m}^{j=2,\dots,\nu}$.

We have the linear system with a matrix $(\nu-1)\times(n+1)$ and with $(\nu-1)(m+1)$ unknown variables. This system may nave nonzero solution only if $n \geq m$, q.e.d.

Remark: The structure of the system for the consecutive set $j_1 = 0,\dots,j_{m+1} = m$ is the following:

Denote

$$u_{k,i}^j(a_l) := \int_{a_1}^{a_l} W^{k+i} W^{(j)},$$

$$u^j(a_l) := \left(u_{k,i}^j(a_l)\right)_{k,i=0,\dots,m},$$

$$c^j = \begin{pmatrix} c_1^j \\ c_2^j \\ \vdots \\ c_m^j \end{pmatrix}.$$

The conditions $\displaystyle\int_{a_1}^{a_l} W^k q = 0$ for $k = 0,\dots,m$ are equivalent to the system

$$u^2(a_l)c^1 + \dots + u^\nu(a_l)c^\nu = 0,$$

and finally we get the system

$$\begin{pmatrix} u^2(a_2) & \cdots & u^\nu(a_2) \\ \vdots & \ddots & \vdots \\ u^2(a_\nu) & \cdots & u^\nu(a_\nu) \end{pmatrix} \begin{pmatrix} c^2 \\ \vdots \\ c^\nu \end{pmatrix} = 0.$$

Remark: We can state a conjecture: the determinant of this matrix is equal to $Const[\prod_{n\geq i>j\geq 1}(a_i-a_j)]^k$.

Theorem 3.4. *Let*

$$P(x)=W(x)=(x-a_1)^{k_1}\cdots(x-a_k)^{k_\nu},$$

$a_1=0$, $a_i\neq a_j$ *for* $i\neq j$, $\deg P=\sum_{i=1}^{\nu}k_i=k$.

Let Q *be a polynomial of degree* $Q=k(m+1)-\alpha$, $\alpha=0,\dots k-1$.

Denote $V_j(x)=\int_{a_1}^{x}P^j(t)q(t)dt$. *Then the minimal necessary number* N *of (not necessary consecutive) conditions* $V_j(x)=0$ *for* $x=a_2,\dots,a_k$ *for the conclusion* $Q=\tilde{Q}(W)$ *for a certain polynomial* $\tilde{Q}$ *without free term is the minimal integer which is greater or equal than* $\frac{(m+1)(k-1)}{\nu-1}$. *For any number of conditions which is less than* N *there exists* Q *unrepresentable as a composition with* W, *for which all* $V_j(a_2),\dots,V_j(a_k)$ *listed above are zeroes.*

Proof of the theorem 3.4.

Assume that $\int_{a_1}^{a_k}P^jq=0$, for $j=j_1,\dots,j_N$, $k=2,\dots,\nu$.

For

$$q(x)=W^m(c_m^1W'+c_m^2W''+\ldots+c_m^kW^{(k)})+\ldots+(c_0^1W'+c_0^2W''+\ldots+c_0^kW^{(k)})$$

we get

$$\int_{a_1}^{a_l}W^kq=\sum_{i=0}^{m}\left(\sum_{j=2}^{k}c_i^j\int_{a_1}^{a_l}W^{k+i}W^{(j)}\right)$$

since the coefficients of the form $\int_0^a W^kW'$ are equal to zero.

As before, we get the linear system with $(m+1)(k-1)$ unknown variables and the matrix of the size $N\times(\nu-1)$. This system may have nonzero solution only if the number of equations is less or equal than the number of unknowns, so $N(\nu-1)\geq(m+1)(k-1)$, q.e.d.

Theorem 3.5. *Let*

$$W(x) = (x - a_1)^{k_1} \cdots (x - a_k)^{k_\nu},$$

$a_1 = 0$, *$a_i \neq a_j$ for $i \neq j$*, $\deg W = \sum_{i=1}^{\nu} k_i = k$, *$P(x) = \gamma_n W^n + \ldots + \gamma_1 W$,* $\deg P = kn$.

Let Q be a polynomial $\deg Q = \nu(m+1) - \alpha$, *$\alpha = 0, \ldots \nu - 1$.*
Denote $V_j(x) = \displaystyle\int_{a_1}^{x} P^j(t)q(t)dt$. Then the minimal necessary number of (not necessary consecutive) conditions $V_j(x) = 0$ for $x = a_2, \ldots, a_k$ for the conclusion $Q = \tilde{Q}(W)$ for a certain polynomial $\tilde{Q}$ without free term is $m+1$. For any number of conditions which is less than $m+1$ there exists Q unrepresentable as a composition with W, for which all $V_j(a_2), \ldots, V_j(a_k)$ listed above are zeroes.

Proof of the theorem 3.5.

We can notice, that $P^s = \displaystyle\sum_{l=0}^{skn} c_l^s W^l$. Then we must simply repeat what was said in each of the previous theorems.

Chapter 4

Maximal number of surviving zeroes.

4.1 Connection between the first model problem and the main problem. A convenient basis for the ideals $I_k, k = 2, \ldots 6$.

The computations in the subsection 4.1 are taken from [2].

Direct computations (including several integrations by part) give the following expressions for the first polynomials $\psi_k(x)$, solving the recurrence relation (1.4):

$$
\begin{aligned}
\psi_2(x) &= -P(x) \\
\psi_3(x) &= P^2(x) - Q(x) \\
\psi_4(x) &= -P^3(x) + 3P(x)Q(x) - \int_0^x q(t)P(t)dt \\
\psi_5(x) &= P^4(x) - 6P^2(x)Q(x) - \int_0^x q(t)P^2(t)dt \\
&\quad +4P(x)\int_0^x q(t)P(t)dt + \frac{3}{2}Q^2(x) \\
\psi_6(x) &= -P^5(x) + 10P^3(x)Q(x) + 5P(x)\int_0^x q(t)P^2(t)dt \\
&\quad -8Q^2(x)P(x) - 10P^2(x)\int_0^x q(t)P(t)dt + 4Q(x)\int_0^x q(t)P(t)dt
\end{aligned}
$$

$$-\int_0^x q(t)P^3(t)dt + \frac{1}{2}\int_0^x p(t)Q^2(t)dt$$

Consequently, we get the following set of generators for the ideals $\tilde{I}_k$, $k = 2,\ldots,6$,

$$\begin{aligned}
I_2 &= \{P\} \\
I_3 &= \{P, Q\} \\
I_4 &= \{P, Q, \int qP\} \\
I_5 &= \{P, Q, \int qP, \int qP^2\} \\
I_6 &= \{P, Q, \int qP, \int qP^2, \int (qP^3 - \frac{1}{2}pQ^2)\}
\end{aligned}$$

Therefore, if $a \in Y(\tilde{I}_6)$ is a zero of the ideal $\tilde{I}_6$, it must satisfy the following equations:

$$\begin{gathered}
P(a) = 0\ ,\ \ Q(a) = 0 \\
\int_0^a P(t)q(t)dt = 0 \\
\int_0^a P^2(t)q(t)dt = 0 \\
\int_0^a P^3(t)q(t)dt - \frac{1}{2}\int_0^a p(t)Q^3(t)dt = 0
\end{gathered} \tag{4.1}$$

Notice that the third and the fourth equations coincide with the moment equations of the first model problem (with the same $p(x)$ and $Q_0(x) = Q(x)$). The fifth equation contains the corresponding term of the model problem and an additional term, which is (for the first time) nonlinear in Q.

Let us assume now that the set of zeroes of $\tilde{I}_6$ consists of the points $a_1 = 0, a_2, \ldots, a_\nu$, $a_i \neq a_j$. In particular, a_i are common zeroes of P and Q, and we can write

$$P(x) = W(x)P_1(x)$$

$$Q(x) = W(x)Q_1(x)$$

where $W(x) = \prod_{i=1}^{\nu}(x - a_i)$.

Substituting this representation into the last three equations of (4.1) and integrating by parts, we get for $i = 1, \ldots, \nu$,

$$\int_0^{a_i} W^2(p_1 Q_1 - P_1 q_1) = 0$$

$$\int_0^{a_i} W^3 P_1(p_1 Q_1 - P_1 q_1) = 0$$

$$\int_0^{a_i} W^4 P_1^2(p_1 Q_1 - P_1 q_1) - \frac{2}{3}\int_0^{a_i} W^3 Q_1(p_1 Q_1 - P_1 q_1) = 0$$

Here $p_1(x) = P_1'(x)$, $q_1(x) = Q_1'(x)$.

4.2 Maximal number of surviving zeroes.

Theorem 4.1. *Either the number of surviving different zeroes (including* 0*) of* I *is less or equal then* $(\deg P + \deg Q)/3$*, or* P *is proportional to* Q*.*

Proof: Let $P = WP_1$, $Q = WQ_1$, $W = \prod_{i=1}^{k}(x - a_i)$ -all surviving zeroes, $\deg P_1 = p$, $\deg Q_1 = q$. Consider the function $f(x) = \int_0^x W^2(p_1 Q_1 - q_1 P_1)dt$ and assume first that $p_1 Q_1 - q_1 P_1 \neq 0$. Then $f(x) = W^3 S(x)$, hence $\deg f(x) \geq 3k$. From the other side $\deg f(x) = 2k+(p+q-1)+1 = p+q+2k$. So, $p + q + 2k \geq 3k$.

Now let $p_1 Q_1 - q_1 P_1 = 0$, i.e. $(P_1 Q_1)' = 2q_1 p_1$. Denote $P_1 Q_1$ by X, Q_1 by Y. Then $q_1 = Y'$, $P_1 = X/Y$, hence $X' = 2Y'\frac{X}{Y}$, i.e. $\frac{X'}{X} = 2\frac{Y'}{Y}$, i.e. $X = CY^2$, i.e. $P_1 Q_1 = CQ_1^2$, q.e.d.

Chapter 5

Verification of the main conjecture.

5.1 The first note about rescaling of P and Q.

As it was shown in [2], it is possible, using rescaling $x \mapsto C_1 x$, $y \mapsto C_2 y$, to make the leading coefficients of P, Q being equal to any positive number. It can be done if $\deg Q \neq 2 \deg P$, but these cases will be not considered in this article. So we will use polynomials P and Q in the form where the leading coefficient equals either 1 or 2, i.e. if required we will be able to deal with polynomials in the form (for $W(x) = x(x-a)$):

$$R(x) = W^k W' + \beta_k + W^{k-1}(\alpha_{k-1} W' + \beta_{k-1}) + ... + (\alpha_0 W' + \beta_0),$$

or

$$R(x) = W^k + W^{k-1}(\alpha_{k-1} W' + \beta_{k-1}) + ... + (\alpha_0 W' + \beta_0).$$

In what follows we shall assume that the highest degree coefficient is not zero. For instance, for the case deg P=3, deg Q=4 we will assume that $P(x) = 2x^3 +$ ... (terms of degree less than 3), $Q(x) = x^4 +$ ...(terms of degree less than 4) and so on.

5.2 The second note about rescaling of P and Q.

Theorem 5.1. *$I = \bigcup_{k=2}^{\infty} I_k$ generated by P, Q has zeroes $a_1, \ldots, a_n$ if and only if $I = \bigcup_{k=2}^{\infty} I_k$ generated by $\lambda^2 P$, λQ $(\lambda \neq 0)$ has the same zeroes.*

Proof of the theorem 5.1.

Notice, that from the definition $\psi_k(x) = Const \int p\psi_{k-1} + Const \int q\psi_{k-2}$, and by induction

$$
\begin{array}{ll}
\psi_0(x) = & 0 \\
\psi_1(x) = & 1 \\
\psi_2(x) = & C_{21} \int p \\
\psi_3(x) = & C_{31} \int p \int p + C_{32} \int q \\
\psi_4(x) = & C_{41} \int p \int p \int p + C_{42} \int p \int q + C_{43} \int q \int p \\
\psi_5(x) = & C_{51} \int p \int p \int p \int p + C_{52} \int p \int p \int q + C_{53} \int p \int q \int p + C_{54} \int q \int p \int p + \\
 & C_{55} \int q \int q \\
\cdots & \cdots \quad ,
\end{array}
$$

where C_{ij} are numerical constants. The statement of the theorem follows from the

Claim. In the expansion of $\psi_k(x)$ are those and only those integrals of the form $\displaystyle\int p \int q \cdots \int q$, for which 2 times the number of $\int q$ + the number of $\int p$ is equal to $k-1$.

Now it is obvious, that if we replace p by $\lambda^2 p$, q by λq, then $\psi_k(x)$ will be multiplied by λ^{k-1}, and hence the zeroes of $\psi_k(x)$ will not change. The theorem is proved.

Proof of Claim.

Induction on k.

Assume the statement is true for $k-1$, k. Let us prove that it is true for $k+1$. Denote by N_k^q the number of $\int q$, by N_k^p – the number of $\int p$. In expansion of $\psi_k(x)$ there are integrals which appear from $\int p\psi_k$, and for them $N_{k+1}^q = N_k^q$, $N_{k+1}^p = N_k^p + 1$, hence $2N_{k+1}^q + N_{k+1}^p = 2N_k^q + N_k^p + 1 = k - 1 + 1 =$

k, q.e.d. For integrals which appear from $\int q\psi_{k-1}$ we get $N^q_{k+1} = N^q_{k-1} + 1$, $N^p_{k+1} = N^p_{k-1}$, hence $2N^q_{k+1} + N^p_{k+1} = 2N^q_{k-1} + 2 + N^p_{k-1} = k - 2 + 2 = k$, q.e.d

Conversely, if we consider any integral of the form $\displaystyle\int p \int q \int p \cdots \int q$ with $2N^q_k + N^p_k = k$, then we can uniquely define an "appearance" of this integral :

$$\begin{array}{rl} \int p \int q \int p \cdots \int q & \in \psi_k(x) \\ \int q \int p \cdots \int q & \in \psi_{k-1}(x) \\ \int p \cdots \int q & \in \psi_{k-3}(x) \end{array}$$

and so on.

5.3 Verification of the main conjecture for the case deg P=3, deg Q=4

The goal of this section is to prove that in this case $I = \displaystyle\bigcup_{k=1}^{\infty} I_k$ can not have zeroes, others then 0. The greater common divisor of 3 and 4 is equal to 1, hence in this case we can not represent $P(x) = \tilde{P}(W(x))$, $Q(x) = \tilde{Q}(W(x))$ with $\deg W \geq 1$ (here $W(x)$ is polynomial, accumulating common zeroes), so the conjecture for this case is true.

1) From the theorem 4.1. we get that the maximal number of surviving zeroes is 2. And one of them is necessarily 0.

2) Assume that I has zeroes 0, a, $(a \neq 0)$. Since zeroes of I should be also zeroes of P and Q , P and Q are necessary represented in the form (up to rescaling)

$$P = WP_1 \ , \ Q = WQ_1,$$

where

$$W = x(x-a) \ , \ \deg P_1 = 1 \ , \ \deg Q_1 = 2.$$

For such P, Q numbers 0, a will be common zeroes of ideals I_1, I_2, I_3. Now represent

$$P_1 = W' + \alpha \ , \ Q_1 = W + \beta W' + \gamma.$$

Then we will directly calculate, using the "Mathematica" software, ideals I_4, I_5, I_6, I_7, I_8 and we will show that for all α, β, γ they can not have zeroes 0,

a. It will complete verification of the main conjecture (1.4) for this case.

3) We will calculate consecutively $\psi_k(a)$, using the following "Mathematica" program:

```
     n=Input[];
% n-the number of ideals to be computed %
     W=x(x-a);
     W'=2x-a;
     P=W*(W'+al);
     Q=W*(W +bt*W' + ga);
     p=D[P,x];
     q=D[Q,x];
     psi[0]=0;
     psi[1]=1;
     psi[2]=-P;
     Do[psi[i]=[Integrate[
         -(i-1)psi[i-1]*p-(i-2)psi[i-2]*q,x],{i,3,n}];
     x=a;
     Do[Print[StringForm["psi[``]=``",i,
          Simplify[psi[i]]]],{i,1,n}];
```

Running this program, we obtain the following results:

$$\psi_4(a) = \frac{-\left(a^5\ (2\,a^2 + 7\,\alpha\,\beta - 7\,\gamma)\right)}{210},$$

$$\psi_5(a) = \frac{a^7\,\alpha\ (a^2 + 3\,\alpha\,\beta - 3\,\gamma)}{315}.$$

Since $a \neq 0$, we get

$$\alpha\beta - \gamma = -2a^2/7\ ,\ \ \alpha(\alpha\beta - \gamma + a^2/3) = 0.$$

It can be satisfied only if $\alpha = 0\ ,\ \gamma = 2a^2/7$. Running the program for these values, we get the following conditions:

$$\psi_6(a) = \frac{a^{11}\ (13 - 21\,a^2)}{4414410},$$

$$\psi_7(a) = \frac{-2\,a^{13}\,\beta}{315315},$$

from which we obtain $\beta = 0$, $a = \pm\sqrt{13/21}$, and for them we get

$$\psi_8(a) = -3668/9,$$

i.e. we obtain contradiction. The conjecture is verified.

5.4 Verification of the main conjecture for the case deg P=4, deg Q=2

The goal of this section is to prove that in this case $I = \bigcup_{k=1}^{\infty} I_k$ has zeroes 0 and a ($a \neq 0$) if and only if $Q(x) = W(x) = x(x-a)$, $P(x) = \tilde{P}(W(x))$ for a certain polynomial $\tilde{P}$ without free term..

Assume that I has zeroes 0, a. Since zeroes of I should be also zeroes of P and Q , P and Q are necessary represented in the form (up to rescaling)

$$P = WP_1 \ , \ Q = W,$$

where $W = x(x-a)$, $\deg P_1 = 2$ For such P, Q numbers 0, a will be common zeroes of ideals I_1, I_2, I_3. Now represent $P_1 = W + \beta W' + \gamma$. Then computing the condition $0 = \psi_4(a) = a^5\beta/30$, we obtain $\beta = 0$, q.e.d.

5.5 Remark about resultants.

Resultants give us a very powerful tool for checking, whether $n+1$ polynomials of n variables $P_i(x_1, \ldots, x_n) \in C[x_1, \ldots, x_n]$ do not have common zeroes.

Consider one example. Assume we are interested whether polynomials $P(x,y)$, $Q(x,y)$, $R(x,y)$ have common zeroes. Compute $\mathbf{Resultant}[P,Q,x] = f_1(y)$, $\mathbf{Resultant}[R,Q,x] = f_2(y)$. If $\mathbf{Resultant}[f_1,f_2,y] \neq 0$, then P, Q, R do not have common zeroes.

Indeed, if there exists common zero of all polynomials (x_0, y_0), then $f_1(y_0) = f_2(y_0) = 0$, hence **Resultant**$[f_1, f_2, y] = 0$, q.e.d.

The general construction is exactly the same.

5.6 Verification of the main conjecture for the case deg P=4, deg Q=3

The goal of this section is to prove that in this case $I = \bigcup_{k=1}^{\infty} I_k$ can not have zeroes, others then 0. The greater common divisor of 4 and 3 is equal to 1, hence in this case we can not represent $P(x) = \tilde{P}(W(x))$, $Q(x) = \tilde{Q}(W(x))$ with $\deg W \geq 1$ (here $W(x)$ is polynomial, accumulating common zeroes), so the conjecture for this case is true.

1) From the theorem 4.1. we get the maximal number of surviving zeroes is 2. And one of them is necessarily 0.

2) Assume that I has zeroes 0, a $(a \neq 0)$. Since zeroes of I should be also zeroes of P and Q , P and Q are necessary represented in the form (up to rescaling)

$$P = WP_1 \ , \ Q = WQ_1,$$

where

$$W = x(x-a) \ , \ \deg P_1 = 2 \ , \ \deg Q_1 = 1.$$

For such P, Q numbers 0, a will be common zeroes of ideals I_1, I_2, I_3. Now represent

$$P_1 = W + \beta W' + \gamma \ , \ Q_1 = W' + \alpha.$$

Then we will directly calculate, using the "Mathematica" software, ideals I_4, I_5, I_6, I_7, I_8 and we will show that for all α, β, γ they can not have zeroes 0, a. It will complete verification of the main conjecture 1.2.7 for this case.

3) We will calculate consecutively $\psi_k(a)$, using the "Mathematica" program, similar to above.

Running the program, we obtain the following results:

$$\psi_4(a) = \frac{a^5 \ (2\,a^2 + 7\,\alpha\,\beta - 7\,\gamma)}{210},$$

$$\psi_5(a) = \frac{a^7\ (4\,a^4 + 11\,a^2\ (\alpha\,\beta - 3\,\gamma) - 66\ (\alpha\,\beta - \gamma)\ \gamma)}{6930}.$$

From the first equation $\alpha\beta = \gamma - 2/7a^2$, substituting into the second equation, we obtain $\gamma = -a^2/77$. So, these and only these conditions force the equations $\psi_4(a) = 0, \psi_5(a) = 0$ to be satisfied. Obviously β may not be equal to zero, so we can put $\alpha = -a^2/77\beta$. Running the program for these values, we obtain the following equations:

$$\psi_6(a) = \frac{a^{11}\ (1573 - 21\,a^4\,\beta + 2541\,a^2\,\beta^3)}{534143610\,\beta},$$

$$\psi_7(a) = \frac{2\,a^{13}\ (-63954\,a^2 + 819\,a^6\,\beta + 3009391\,\beta^2 - 112651\,a^4\,\beta^3)}{948906123165\,\beta},$$

$$\psi_8(a) = (a^{13}(315517059a^2 + 47966683149\beta^2 - 1036350a^{10}\beta^2 - 6465588052a^4\beta^3$$
$$+151367370a^8\beta^4 + 10626a^6\beta(7543 + 93170\beta^5)))/(20166152929502580\beta^2).$$

We obtain three polynomials of two variables a, β. Now according to section 5.5. we can compute

$$\textbf{Resultant}[\psi_6(a), \psi_7(a), \beta] = 171355466545636153516888971819\ a^2$$
$$-553814823359352913561 28\ a^{12} + 4434102226084608\ a^{22},$$
$$\textbf{Resultant}[\psi_6(a), \psi_8(a), \beta] = Const(216908655616510696903575187607931a^8$$
$$+458786962298721610776125188208a^{18} - 161604860797505145100608a^{28}$$
$$+14080788862156800a^{38}),$$

and computing resultant of the last two expressions (dividing by the proper power of a) we get nonzero number, q.e.d.

5.7 Verification of the main conjecture for the case deg P=4, deg Q=4

The goal of this section is to prove that in this case $I = \bigcup_{k=1}^{\infty} I_k$ has common zeroes others than 0 if and only if either $P(x) = \tilde{P}(W(x))$, $Q(x) = \tilde{Q}(W(x))$ for certain polynomials $\tilde{P}$, $\tilde{Q}$ without free terms, where $W(x) = x(x-a)$, $a \neq 0$, or P is proportional to Q (and in this case $W = P$ and again $P = \tilde{P}(W)$, $Q = \tilde{Q}(W)$.

1) Let P, Q be not proportional. From the theorem 4.1. we get the maximal number of surviving zeroes is 2. And one of them is necessarily 0.

2) Assume that I has zeroes 0, a. Since zeroes of I should be also zeroes of P and Q , P and Q are necessary represented in the form (up to rescaling)

$$P = WP_1 \ , \ Q = WQ_1,$$

where

$$W = x(x-a) \ , \ \deg P_1 = 2 \ , \ \deg Q_1 = 2.$$

For such P, Q numbers 0, a will be common zeroes of ideals I_1, I_2, I_3. Now represent

$$P_1 = W + \gamma W' - \alpha \ , \ Q_1 = W + \delta W' - \beta.$$

Then we will directly calculate, using the "Mathematica" software, ideals I_4, I_5, I_6, I_7, I_8 and we will show that the only possibilities for I to have zeroes 0, a are either $\gamma = \delta = 0$ or $P = Q$. It will complete verification of the main conjecture 1.2.7 for this case.

3) We will calculate consecutively $\psi_k(a)$, using the "Mathematica" program, similar to above. Running the program, we obtain the following results:

$$\psi_4(a) = \frac{a^5 \ (7\,\alpha\,\delta + 2\,a^2\ (\delta - \gamma) - 7\,\beta\,\gamma)}{210},$$

$$\psi_5(a) = \frac{a^7\ (4\,a^4\ (\delta-\gamma) + 66\,\alpha\ (\alpha\,\delta - \beta\,\gamma) + 11\,a^2\ (3\,\alpha\,\delta - 2\,\alpha\,\gamma - \beta\,\gamma))}{6930}.$$

Since $a \neq 0$, we get

$$\frac{2}{7}a^2(\delta - \gamma) = \beta\gamma - \alpha\delta,$$

$$(4a^4 + 22a^2\alpha)(\delta-\gamma) + (\alpha\delta - \beta\gamma)(66\alpha + 11a^2) = 0.$$

If $\delta = \gamma$, then from the first equation $\alpha = \beta$, and hence $P = Q$. So, $\delta \neq \gamma$ and dividing the second equation by $\delta - \gamma$, we obtain $\alpha = -3a^2/11$. Then $\delta = 77\beta\gamma/a^2 + 22\gamma$.

Running the program for these values, we get

$$\psi_6(a) = \frac{-\left(a^7\ (3\,a^2 + 11\,\beta)\ \gamma\ (-4719\,a^2 + 3\,a^6 - 17303\,\beta - 363\,a^4\,\gamma^2)\right)}{10900890}.$$

If $\gamma = 0$, then $\delta = 0$, q.e.d.

If $\beta = -3a^2/11$, then $\beta = \alpha$, hence $\delta = \gamma$, and hence $P = Q$.

So,

$$\beta = \frac{3a^6 - 4719a^2 - 363a^4\gamma^2}{17303},$$

and running the program for these values (i.e. without α, δ, β), we get

$$\psi_7(a) = (2a^{15}(a-11\gamma)\gamma(a+11\gamma)(508079a^2+711a^6-61477559\gamma^2-629926a^4\gamma^2$$
$$+99309903a^2\gamma^4))/1452013567609605$$

$$\psi_8(a) = (a^{15}(a-11\gamma)\gamma(a+11\gamma)(-165436111269a^2+23749415118a^6+37532547a^{10}$$
$$+20017769463549\gamma^2-2920126191268a^4\gamma^2-28998322881a^8\gamma^2-64672793651466a^2\gamma^4$$
$$+4252091239473a^6\gamma^4 + 52235717949261a^4\gamma^6))/6303887863505747915 40.$$

$$\psi_9(a) = -(a^{19}(a-11\gamma)\gamma(a+11\gamma)(-166460483475a^2+4591313298a^6+8200347a^{10}$$
$$+22896141543275\gamma^2-436487623572a^4\gamma^2-5747949999a^8\gamma^2-49500160259550a^2\gamma^4$$
$$+782017522209a^6\gamma^4 + 20215061125875a^4\gamma^6)/3467138324928161353470$$

If $\gamma = \pm a/11$, then $\beta = -3a^2/11$, so $\alpha = \beta$ and hence $\gamma = \delta$, so $P = Q$.

If $\gamma = 0$, then $\delta = 0$, q.e.d.

Otherwise we get 3 polynomials in two variables γ, a. Computing resultants, we get nonzero number, q.e.d. The conjecture for this case is completely verified.

5.8 Verification of the main conjecture for the case deg P=5, deg Q=2

The goal of this section is to prove that in this case $I = \bigcup_{k=1}^{\infty} I_k$ can not have zeroes, others then 0. the greater common divisor of 3 and 4 is equal to 1, hence in this case we can not represent $P(x) = \tilde{P}(W(x))$, $Q(x) = \tilde{Q}(W(x))$ with $\deg W \geq 1$ (here $W(x)$ is polynomial, accumulating common zeroes), so the conjecture for this case is true.

1) From the theorem 4.1. we get the maximal number of surviving zeroes is 2. And one of them is necessarily 0.

2) Assume that I has zeroes 0, a, $a \neq 0$. Since zeroes of I should be also zeroes of P and Q , P and Q are necessary represented in the form (up to rescaling)

$$P = WP_1,\ Q = W,$$

where

$$W = x(x-a)\ ,\ \deg P_1 = 3\ .$$

For such P, Q numbers 0, a will be common zeroes of ideals I_1, I_2, I_3. Now we represent $P_1 = WW' + \alpha W + \beta W' + \gamma$. Then we will directly calculate, using the "Mathematica" software, ideals I_4, I_5, I_6, I_7, I_8 and we will show that for all α, β, γ they can not have zeroes 0, a. It will complete verification of the main conjecture 1.2.7 for this case.

3) We will calculate consecutively $\psi_k(a)$, using the "Mathematica" program, similar to above. The condition $I_4 = 0$ is

$$-a^7/210 + a^5\beta/30 = 0,$$

from this we deduce $\beta = a^2/77$. After substitution β we obtain the following condition from $I_5 = 0$: $a^9(-3a^2\alpha + 11\gamma) = 0$, from which $\gamma = \frac{3a^2}{11}\alpha$. After substituting it into the conditions $I_6 = 0$, $I_7 = 0$, $I_8 = 0$ we get

$$\psi_6(a) = -a^9 \frac{-14413399 + 1089a^8 - 2499a^6\alpha^2}{127126179180},$$

$$\psi_7(a) = a^{13}\alpha \frac{-1641486 + 121a^8 - 273a^6\alpha^2}{316302041055},$$

$$\psi_8(a) = a^{11}\frac{91664331497739 - 17523432234a^8 + 732050a^{16}}{1976282987091252840} +$$
$$+a^{11}\frac{-302413186614a^6\alpha^2 + 21130956a^{14}\alpha^2 - 50781150a^{12}\alpha^4}{1976282987091252840}.$$

We obtain three polynomials of two variables a, α. Now according to the section 5.5. we can compute **Resultant**$[\psi_6(a), \psi_7(a), \alpha]$, **Resultant**$[\psi_6(a), \psi_8(a), \alpha]$, and computing resultant of the last two expressions (dividing by the proper power of a) we get nonzero number, q.e.d.

5.9 Verification of the main conjecture for the case deg P=6, deg Q=2

The goal of this section is to prove that in this case $I = \bigcup_{k=1}^{\infty} I_k$ has zeroes 0 and a ($a \neq 0$) if and only if $Q(x) = W(x) = x(x-a)$, $P(x) = \tilde{P}(W(x))$ for a certain polynomial $\tilde{P}$ without free term.

1) Assume that I has zeroes 0, a. Since zeroes of I should be also zeroes of P and Q , P and Q are necessary represented in the form (up to rescaling)

$$P = WP_1\ ,\ Q = W,$$

where $W = x(x-a)$, $\deg P_1 = 2$ For such P, Q numbers 0, a will be common zeroes of ideals I_1, I_2, I_3.

2) Now represent

$$P_1(x) = W^2 + \alpha WW' + \beta W + \gamma W' + \delta.$$

Then we directly calculate, using the "Mathematica" software, ideals I_4, I_5, I_6, I_7, I_8 and we get $\psi_4(a) = 0$ implies $\gamma = a^2\alpha/7$, substituting it into $\psi_5(a) = 0$ we obtain $a^9\alpha(9a^4 - 39a^2\beta + 143\delta) = 0$, and hence either $\alpha = 0$ (and then $\gamma = 0$) and we are done or $\alpha \neq 0$ and then $\delta = -\frac{9a^4-39a^2\beta}{143}$.

Let us prove that in the second case we obtain contradiction. Running the "Mathematica" program for these values, we get

$$\psi_6(a) = \frac{a^9\,\alpha\,(7120219106 - 38367\,a^{10} + 1234506\,a^6\,\beta^2 - 342\,a^8\,(1573\,\alpha^2 + 1127\,\beta))}{62800332514920},$$

$$\psi_7(a) = ...,$$

$$\psi_8(a) = ...,$$

$$\psi_9(a) = ...$$

So, we get 4 equations in 3 variables a, α, β, and using the same algorithm as above we obtain the contradiction, which completes the proof of the conjecture for this case.

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